

ELECTRIC IMPEDANCE OF SUSPENSIONS OF SPHERES.

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I.

The electric resistance of a suspension of homogeneous spheres is given by the relation

$$\frac{r_1/r - 1}{r_1/r + 2} = \rho \frac{r_1/r_2 - 1^1}{r_1/r_2 + 2} \quad (1)$$

in which r , r_1 , and r_2 are the resistances² of the suspension, the suspending, and suspended phases respectively, and ρ is the volume concentration of the suspended phase.

Since emulsions, suspensions of living cells, and colloidal particles have an interfacial surface layer which is markedly different from both the interior and exterior phases, the suspended phase can only be considered homogeneous under a few special conditions. As has been shown by Maxwell (9), §313, a sphere of radius a_2 and resistance r_2 surrounded by a concentric spherical shell of internal radius a_2 ,

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¹ This equation has been variously derived in the early theory of several physical problems but it often disagrees with the data (Lowry (8)) for atomic and molecular phenomena—probably because the assumptions underlying it are not valid in these cases. It has, however, been found by Fricke and Morse (3) to apply accurately to the resistance of cream up to 62 per cent volume concentration of butter fat, and its use is probably justified for the resistance of suspensions of spheres which are of much larger than atomic dimensions. A simple derivation is given by Maxwell (9), §314.

² Unless otherwise stated, all resistances, reactances, and impedances are specific, *i.e.*, for a centimeter cube.

external radius a_3 , and resistance r_3 , Fig. 1, can be replaced by a homogeneous sphere of radius a_3 and equivalent resistance \hat{r}_2 where

$$\hat{r}_2 = r_2 \frac{(2r_2 + r_3) a_3^3 + (r_2 - r_3) a_2^3}{(2r_2 + r_3) a_3^3 - 2(r_2 - r_3) a_2^3}.$$

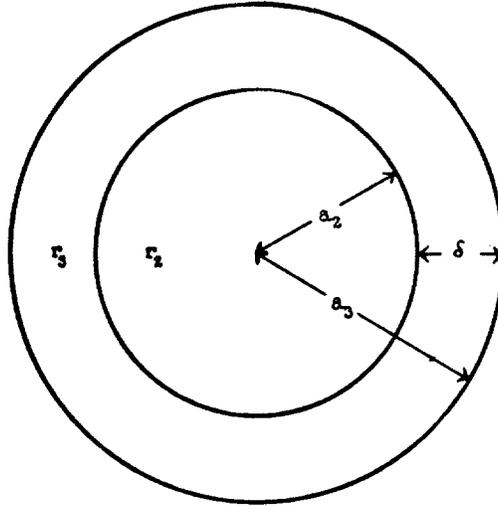


FIG. 1.

When $a_2 = a_3 - \delta = a - \delta$, where δ is a small quantity such that higher powers than the first can be neglected,

$$\hat{r}_2 = \frac{r_2 + \frac{\delta r_3}{a} \left(1 - \frac{r_2}{r_3}\right)}{1 - \frac{2\delta}{a} \left(1 - \frac{r_2}{r_3}\right)}.$$

If further, r_2/r_3 is small compared with unity,

$$\hat{r}_2 = r_2 + \frac{\delta r_3}{a}. \quad (2)$$

Assuming that the current flow in the surface layer is radial, then δr_3 , which is a resistance per unit area, may be replaced by z_3 , a complex

impedance per unit area given in alternating current vector notation (1, 7, 11) by

$$z_2 = r_2 + jx_2 \quad (3)$$

where the resistance per unit area r_2 is in series with a reactance per unit area x_2 and j is the imaginary operator. Substituting z_2 of Eq. (3) for δr_2 in Eq. (2) and replacing \hat{r}_2 by \hat{z}_2 the equivalent complex impedance of the sphere, we have

$$\hat{z}_2 = r_2 + z_2/a = r_2 + r_2/a + jx_2/a. \quad (4)$$

Solving Eq. (1) for r ,

$$r = r_1 \frac{(1 - \rho) r_1 + (2 + \rho) r_2}{(1 + 2\rho) r_1 + 2(1 - \rho) r_2}.$$

Substituting \hat{z}_2 for r_2 and replacing r by z , the complex impedance of the suspension,

$$z = r_1 \frac{(1 - \rho) r_1 + (2 + \rho) (r_2 + r_2/a) + (2 + \rho) jx_2/a}{(1 + 2\rho) r_1 + 2(1 - \rho) (r_2 + r_2/a) + 2(1 - \rho) jx_2/a}. \quad (5)$$

On the assumption that r_1 and r_2 are constant, when $z_2 \rightarrow \infty$, z is a pure resistance

$$r_0 = r_1 \frac{2 + \rho}{2(1 - \rho)}; \quad (6)$$

and when $z_2 \rightarrow 0$, z is again a pure resistance

$$r_\infty = r_1 \frac{(1 - \rho) r_1 + (2 + \rho) r_2}{(1 + 2\rho) r_1 + 2(1 - \rho) r_2}. \quad (7)$$

These are two cases in which the spheres may be considered homogeneous.

Solving Eq. (5) for z_2

$$z_2 = \frac{z - r_\infty}{r_0 - z} \gamma \quad (8)$$

where

$$\gamma = r_2 a \left[1 + \frac{(1 + 2\rho)}{2(1 - \rho)} \cdot \frac{r_1}{r_2} \right]. \quad (9)$$

When $r_2 = 3r_1$, and ρ varies from 0.2 to 0.4, the second term in the brackets varies from 0.3 to 0.5. Thus γ is of the same order of magnitude as $r_2 a$ and more or less constant for many cases.

Separating z into its resistance and reactance components,

$$z = r + jx,$$

the absolute value or magnitude of $|z_3|$ is given by

$$|z_3|^2 = \frac{(r - r_\infty)^2 + x^2}{(r_0 - r)^2 + x^2} \gamma^2. \quad (10)$$

In the special case where r_3 is constant or zero, it can be shown that

$$(r - r_\infty)(r_0 - r) = x^2.$$

Then

$$\frac{x_3^2}{\gamma^2} = \frac{r - r_\infty}{r_0 - r}, \quad (11)$$

and also

$$\frac{x_3^2}{\gamma^2} = \frac{|z|^2 - r_\infty^2}{r_0^2 - |z|^2}. \quad (12)$$

II.

When r_2 is finite and z_3 varies with the frequency n of the measuring current so that when $n \rightarrow 0$, $z_3 \rightarrow \infty$, and when $n \rightarrow \infty$, $z_3 \rightarrow 0$, then $|z|$ from Eq. (5) as a function of n is shown in Fig. 2.

If r_3 varies such that

$$r_3 = m x_3 \quad (13)$$

then it can be shown that as x_3 varies from 0 to ∞ the locus of the end of the impedance vector when plotted on the complex plane is an arc of a circle as shown in Fig. 3. Also the ratio of the chords a and b is

$$\frac{a}{b} = \frac{|z_3|}{\gamma}. \quad (14)$$

At the points r_∞ and r_0 , the slopes of the tangents are respectively $1/m$ and $-1/m$.

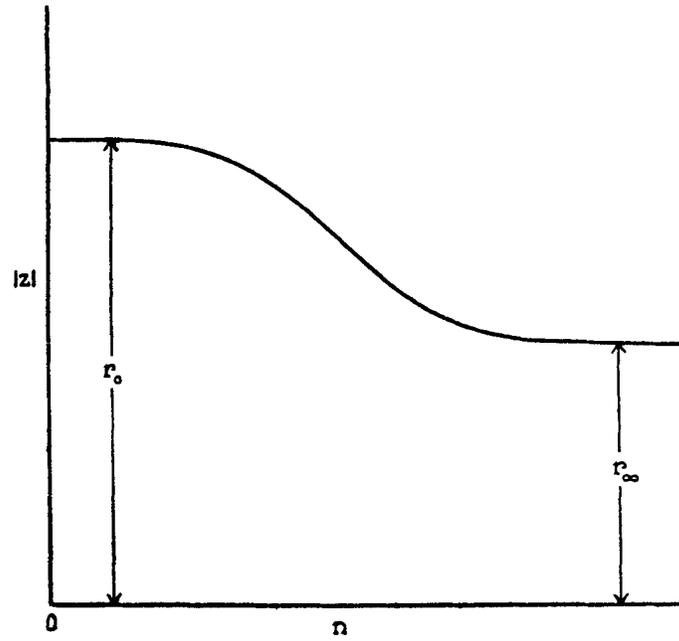


FIG. 2.

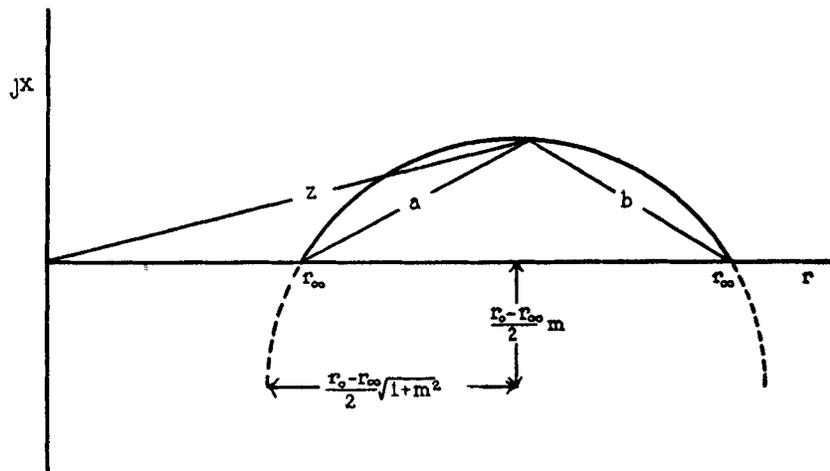


FIG. 3.

The case where $m = 0$, which gives a circle with its center on the resistance axis, has been treated in the first part of a paper by Carter (2). It will be noted also that his Eq. (10), for the case of a single reactance in a resistance network, is, as it should be, the exact counterpart of Eq. (5), above, when r_3 is constant.

III.

Zobel (13) has shown that certain types of two terminal networks—of which the circuits of Fig. 4 are special cases—can be made equivalent both in impedance and in phase angle for all frequencies. As a result

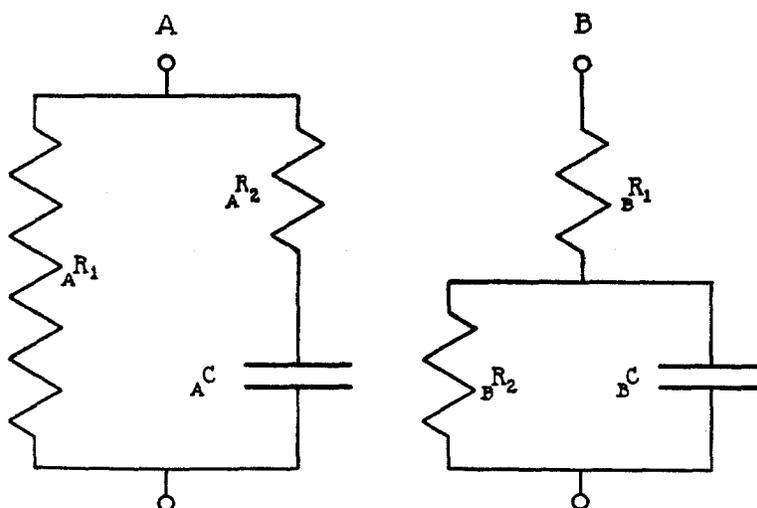


FIG. 4.

of this, such circuits containing any number of resistances and a capacity, can be made equivalent to either one of the two simple circuits. Thus it is evident that the number, location, and magnitude of the elements of such a circuit cannot be determined solely by electrical measurements made at the terminals, and that the number of circuits which can be made to fit a given set of data is probably limited only by the patience and ingenuity of the computer.

Fricke and Morse (6) found that their measurements of the resistance and capacity of suspensions of red blood cells at various frequen-

cies could be accurately fitted to a circuit of type A , where they thought of ${}_A R_1$ as due to the suspending medium, ${}_A R_2$ to the interiors of the corpuscles, and ${}_A C$ to the capacities at their surfaces. The values of these three quantities can be found for a suspension of spheres from Eq. (5) when $r_3 = 0$ and $x_3 = -\frac{1}{c_3 \omega}$, ω being 2π times the frequency in cycles per second. In this case ${}_A R_1$ depends on r_1 , and ρ , ${}_A R_2$ on r_1 , as well as on r_2 and ρ , while

$${}_A C = \frac{1}{2} (2 + r_1/r_0) (1 - r_1/r_0) a c_3.$$

For small volume concentrations this becomes approximately

$${}_A C = \frac{3}{2} (1 - r_1/r_0) a c_3, \quad (15)$$

which is the expression derived by Fricke (4) for spheres by a quite different analysis. Circuits of type B have been computed which fit the data, but they have no particular interest since Eq. (15) was derived on the basis of type A .

In his work on the impedance of tissues Philippon (10) assumed a circuit of type B where ${}_B R_2$ and ${}_B C$ represented the resistances and capacities of the physiological cell membranes, and ${}_B R_1$ the resistances of the protoplasm of the cells in a centimeter cube of tissue. The data have been used equally well to compute circuits of type A , in which ${}_A R_1$ might be thought to be due to the intercellular electrolytes, ${}_A C$ to the membrane capacities, while ${}_A R_2$ involves the resistances of the protoplasm. Thus the interpretation of these data should not be made intuitively.

Whereas Fricke (5) found the red blood cell membrane to have a static capacity (*i.e.* independent of frequency), Philippon (10) found a capacity which for animal tissues varied about as the inverse square root of the frequency, and for vegetable tissues as the inverse fourth root. Although unable to measure the magnitude of these capacities for a single membrane, Philippon classed them as polarization capacities similar to those found at the surface of contact between metal electrodes and electrolytes. Such physical systems have a polarization resistance such that the phase angle, ϕ_3 , of the

combination is more or less independent of frequency and often in the neighborhood of 45° (see Wolff (12)). Then in Eq. (13), $\phi_3 = \text{ctn}^{-1} m$. On the assumption that this angle was 45° , Philippson calculated the series polarization resistance although he took no account of it in the computations of the capacities from the data. In Eq. (5), the polarization resistance would be represented by r_3 as a series resistance per unit area. Since r_3 would be a function of frequency, more or less proportional to x_3 , it would be difficult if not impossible to compute it from impedance data alone without making assumptions which are at present unwarranted. If, however, measurements are made of both the resistance and reactance of a suspension—with a capacity bridge, for instance—it becomes possible to determine both r_3 and x_3 from Eq. (5).

SUMMARY.

A general expression has been derived for the electric impedance of a suspension of spheres each having a homogeneous non-reactive interior and a thin surface layer with both resistance and reactance. The applications and limitations of impedance measurements on such suspensions are discussed.

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